Stochastic Calculus for Finance II
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some Solutions to Chapter IX

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Exercise 9.3 (Change in Volatility caused by Change of Numéraire)

(i) Let $f(x,y) = \frac{x}{y}$ such that the non-zero derivatives are

$$\frac{\partial f}{\partial x} = \frac{1}{y}, \quad \frac{\partial f}{\partial y} = -\frac{x}{y^2}, \quad \frac{\partial^2 f}{\partial y^2} = \frac{2x}{y^3}, \quad \frac{\partial^2 f}{\partial x \partial y} = -\frac{1}{y^2}.$$

The differential of $S^{(N)}(t) = f(S(t), N(t))$ is then given by

$$\begin{aligned} d\left(S^{(N)}(t)\right) &= \frac{1}{N(t)}dS(t) - \frac{S(t)}{N^2(t)}dN(t) - \frac{1}{N^2(t)}dS(t)dN(t) - \frac{S(t)}{N^3(t)}\left(dN(t)\right)^2 \\ &= \sigma S^{(N)}d\tilde{W}_1(t) - \nu S^{(N)}d\tilde{W}_3(t) - \sigma\nu\rho S^{(N)}dt + \nu^2 S^{(N)}dt \\ \frac{d\left(S^{(N)}(t)\right)}{S^{(N)}} &= \left(\nu^2 - \sigma\nu\rho\right)dt + \sigma d\tilde{W}_1(t) - \nu d\tilde{W}_3(t). \end{aligned}$$

We now want to check if it is possible to find a $\gamma \in \mathbb{R}$ such that $\gamma\tilde{W}_4(t) = \sigma\tilde{W}_1(t) - \nu\tilde{W}_3(t)$ and $\tilde{W}_4(t)$ is a Brownian motion. We first note that $\sigma\tilde{W}_1(t) - \nu\tilde{W}_3(t)$ is a continuous martingale, starting at zero in $t = 0$ and has zero expected value. Its quadratic variation is

$$\left(\sigma d\tilde{W}_1(t) - \nu d\tilde{W}_3(t)\right)^2 = (\sigma^2 + \nu^2 - 2\sigma\nu\rho)dt.$$

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Now let

$$\gamma = \frac{1}{\sqrt{\sigma^2 dt - 2\sigma\nu\rho dt + \nu^2}}$$

and define

$$\tilde{W}_4(t) = \frac{\sigma d\tilde{W}_1(t) - \nu d\tilde{W}_3(t)}{\sqrt{\sigma^2 dt - 2\sigma\nu\rho dt + \nu^2}}.$$

Note that $d\tilde{W}_4(t)d\tilde{W}_4(t) = dt$ and by Lévy's theorem $\tilde{W}_4(t)$ is a Brownian motion.

We can write

$$\frac{d\left(S^{(N)}(t)\right)}{S^{(N)}(t)} = (\nu^2 - \sigma\nu\rho)dt + \gamma d\tilde{W}_4(t).$$

(ii) In order for $\tilde{W}_2(t)$ to be a Brownian motion, we require $\left(d\tilde{W}_2(t)\right)^2 = dt$. Our second condition is $d\tilde{W}_1(t)d\tilde{W}_2(t) = 0$, which comes from the independence of $\tilde{W}_1(t)$ and $\tilde{W}_2(t)$. We search for $a, b \in \mathbb{R}$ such that

$$\tilde{W}_2(t) = a\tilde{W}_1(t) + b\tilde{W}_3(t)$$

and both conditions are fulfilled. We start by calculating the cross variation between $\tilde{W}_1(t)$ and $\tilde{W}_2(t)$

$$d\tilde{W}_1(t)d\tilde{W}_2(t) = (a + b\rho)\, dt.$$

This term is zero if $a = -b\rho$. The quadratic variation of $\tilde{W}_2(t)$ is

$$\left(d\tilde{W}_2(t)\right)^2 = \left(a^2 + 2ab\rho + b^2\right) dt.$$

Using $a^2 + 2ab\rho + b^2 = 1$ and $a = -b\rho$ we get

$$b = \pm\frac{1}{\sqrt{1-\rho^2}}, \quad a = \mp\frac{\rho}{\sqrt{1-\rho^2}}.$$

It follows that

$$\tilde{W}_2(t) = \mp\frac{\rho}{\sqrt{1-\rho^2}}\tilde{W}_1(t) \pm \frac{1}{\sqrt{1-\rho^2}}\tilde{W}_3(t).$$

Note that both solutions are equivalent due to the symmetry of the Brownian motion. We thus only consider the first one in the following. Solving for $\tilde{W}_3(t)$ and substituting into the SDE for $N(t)$ yields

$$dN(t) = rN(t)dt + \nu N(t)\left[\rho d\tilde{W}_1(t) + \sqrt{1-\rho^2}d\tilde{W}_2(t)\right] \qquad \text{(q.e.d)}$$

(iii) Using the result from part (ii), the differentials of the discounted asset prices can be written as

$$\begin{aligned} d\left(D(t)S(t)\right) &= D(t)S(t)\sigma d\tilde{W}_1(t) \\ d\left(D(t)N(t)\right) &= D(t)N(t)\left[\nu\rho d\tilde{W}_1(t) + \nu\sqrt{1-\rho^2}d\tilde{W}_2(t)\right]. \end{aligned}$$

The volatility vectors are given by

$$\sigma = \begin{bmatrix} \sigma \\ 0 \end{bmatrix}, \qquad \nu = \begin{bmatrix} \nu\rho \\ \nu\sqrt{1-\rho^2} \end{bmatrix}.$$

By Theorem 9.2.2, the volatility under the numeraire measure becomes

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} \sigma - \nu\rho \\ -\nu\sqrt{1-\rho^2} \end{bmatrix}$$

and we have

$$\sqrt{v_1^2 + v_2^2} = \sqrt{\sigma^2 - 2\sigma\nu\rho + \nu^2} \qquad \text{(q.e.d.)}$$

Exercise 9.5 (Quanto Option)

We first give a general solution to the problem in exercise (i) and (ii), i.e. finding the solution to a (multidimensional) geometric Brownian motion. Let $S(t)$ be an asset as defined in the mutlidimensional market model in Equation (5.4.2), i.e.

$$dS(t) = \alpha(t)S(t)dt + S(t)\sum_{j=1}^{d}\sigma_j(t)dW_j(t).$$

Now let $f(t,x) = \ln x$. We have

$$\frac{\partial f}{\partial t} = 0, \quad \frac{\partial f}{\partial x} = \frac{1}{x} \quad \frac{\partial^2 f}{\partial x^2} = -\frac{1}{x^2}.$$

Since the Brownian motions are assumed to be independent, we have $dW_j(t)dW_k(t) = 0$ for $j \neq k$ and thus

$$(dS(t))^2 = S^2(t) \sum_{j=1}^{d} \sigma_j^2(t)dt = \|\sigma(t)\|^2 \, dt.$$

Applying Itô's lemma yields the differential of the logarithmic asset price as

$$\begin{aligned} d \ln S(t) &= \frac{1}{S(t)} dS(t) - \frac{1}{2} \frac{1}{S^2(t)} (dS(t))^2 \\ &= \left(\alpha(t) - \frac{1}{2} \|\sigma(t)\|^2 \right) dt + \sum_{j=1}^{d} \sigma_j(t) dW_j(t). \end{aligned}$$

We integrate to obtain

$$\ln S(t) = \ln S(o) + \int_0^t \left(\alpha(s) - \frac{1}{2} \|\sigma(s)\|^2 \right) ds + \int_0^t \sum_{j=1}^{d} \sigma_j(s) dW_j(s).$$

Finally, taking the exponential yields

$$S(t) = S(0) \exp \left\{ \int_0^t \left(\alpha(s) - \frac{1}{2} \|\sigma(s)\|^2 \right) ds + \int_0^t \sum_{j=1}^{d} \sigma_j(s) dW_j(s) \right\}.$$

and in case of constant drift and diffusion coefficients $\alpha(t) = \alpha$ and $\sigma(t) = \sigma$, we get

$$S(t) = S(0) \exp \left\{ \left(\alpha - \frac{1}{2} \|\sigma\|^2 \right) t + \sum_{j=1}^{d} \sigma_j W_j(t) \right\}.$$

(i) We have $\alpha(t) = r$, $\sigma(t) = \sigma_1$. By the previous analyses, $S(t)$ is given by

$$S(t) = S(0) \exp \left\{ \left(r - \frac{1}{2} \sigma_1^2 \right) t + \sigma_1 d\tilde{W}_1(t) \right\}. \qquad \text{(q.e.d.)}$$

(ii) We have $\alpha(t) = r - r^f$, $\sigma(t) = \begin{pmatrix} \rho\sigma_2 & \sqrt{1-\rho^2}\sigma_2 \end{pmatrix}^T$. Since $\|\sigma(t)\|^2 = \sigma_2^2 \rho^2 + \sigma_2^2 (1 - \rho^2) = \sigma_2^2$, we obtain

$$Q(t) = Q(0) \exp \left\{ \left(r - r^f - \frac{1}{2} \sigma_2^2 \right) t + \rho\sigma_2 \tilde{W}_1(t) + \sqrt{1-\rho^2} \sigma_2 \tilde{W}_2(t) \right\}. \qquad \text{(q.e.d.)}$$

(iii) We have

$$\frac{S(t)}{Q(t)} = \frac{S(0)}{Q(0)} \left\{ \left(r^f - \frac{1}{2}\sigma_1^2 + \frac{1}{2}\sigma_2^2 \right) t + (\sigma_1 - \rho\sigma_2)\, \tilde{W}_1(t) - \sqrt{1-\rho^2}\sigma_2 \tilde{W}_2(t) \right\}.$$

We want to find a σ_4 such that $\tilde{W}_4(t)$ is a Brownian motion and

$$\sigma_4 \tilde{W}_4(t) = (\sigma_1 - \rho\sigma_2)\, \tilde{W}_1(t) - \sqrt{1-\rho^2}\sigma_2 \tilde{W}_2(t).$$

By Lévy's theorem, $\tilde{W}_4(t)$ is a Brownian motion if it is a continuous martingale starting a $\tilde{W}_4(0) = 0$ and with unit quadratic variation. The continuity, martingale and initial value properties directly follows from the definition of $\tilde{W}_4(t)$ as the sum of two Brownian motions. The quadratic variation is

$$d\tilde{W}_4(t) d\tilde{W}_4(t) = \frac{(\sigma_1 - \rho\sigma_2)^2 - (1-\rho^2)\,\sigma_2^2}{\sigma_4^2} dt = \frac{\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2}{\sigma_4^2} dt.$$

We thus set $\sigma_4 = \sqrt{\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2}$ such that $\tilde{W}_4(t)$ is a $\tilde{\mathbb{P}}$ Brownian motion. Substituting in the formula for the currency converted spot price yields

$$\begin{aligned} \frac{S(t)}{Q(t)} &= \frac{S(0)}{Q(0)} \left\{ \left(r^f - \frac{1}{2}\sigma_1^2 + \frac{1}{2}\sigma_2^2 + \frac{1}{2}\sigma_4^2 - \frac{1}{2}\sigma_4^2 \right) t + \sigma_4 d\tilde{W}_4(t) \right\} \\ &= \frac{S(0)}{Q(0)} \left\{ \left(r^f - \rho\sigma_1\sigma_2 + \sigma_2^2 - \frac{1}{2}\sigma_4^2 \right) t + \sigma_4 d\tilde{W}_4(t) \right\} \\ &= \frac{S(0)}{Q(0)} \left\{ \left(r - a - \frac{1}{2}\sigma_4^2 \right) t + \sigma_4 d\tilde{W}_4(t) \right\}. \end{aligned}$$

In the last step, we defined $a = r - r^f + \rho\sigma_1\sigma_2 - \sigma_2^2$.

(iv) It follows that the problem of pricing a quanto call option is equivalent to the one of pricing a plain vanilla call option when the underlying pays a continuous dividend yield of a. The solution for this case is given in Section 5.5.2 in Equations (5.5.11) and (5.5.12) and is not repeated here.